ON THE DRIVING OF A PISTON WITH A RIGID COLLAR INTO AN ELASTIC HALF-SPACE

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This paper is concerned with the study of the problem of a field of steady-state vibrations excited in an elastic half-space by means of a rigid circular piston with an infinite rigid and smooth collar. Formulas for the active and reactive resistance of the connection between the field of wave propagation and the piston are obtained in terms of tabulated functions. Results of the analysis are presented for the case of driving a piston into an elastic Poisson medium.

Let us study an elastic half-space $z > 0$. A circular piston of radius c is oscillating harmonically according to the law $dz/dt = v \exp(i\omega t)$ in the opening of a rigid collar at the surface of the half-space. Contact of the piston with the elastic medium is assured during the entire cycle of the oscillation by some constant load, such that the piston executes small oscillations about the position of equilibrium. Because of the principle of superposition of states of stress in the linear theory of elasticity the static field and the field of steady-state oscillations are independent.

The analysis of the latter reduces to the solution of the dynamic equations of the theory of elasticity with the condition that the normal displacements at the points under the rigid piston are equal to $z = (v/i\omega)\exp(i\omega t)$, where v is the amplitude of the velocity of the piston. At points under the rigid collar the normal displacements are equal to zero. In calculations where the dimensions are large, the collar can be assumed to be infinite. The surface of the piston and the collar will be also assumed to be sufficiently smooth, such that the shear stresses that might arise on the surface of the medium can be neglected.

The mathematical problem of the analysis of the wave field, which depends on time as $\exp(i\omega t)$, reduces to finding bounded solutions of the equations on the amplitudes of the scalar displacement potential ϕ and the angular component of the vector displacement potential ψ. There is only a single angular component of ψ because of the symmetry of the problem. The equations

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\partial^2\varphi}{\partial z^2}+k_a{}^2\varphi=0 \tag{1}$$

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{\psi}{r^2}+\frac{\partial^2\psi}{\partial z^2}+k_b{}^2\psi=0 \tag{2}$$

are to be solved for the boundary conditions representing the given values of the shear stresses τ_{rz} and normal displacements u_z at the surface:

$$\frac{1}{\mu}\tau_{rz}\,|_{z=0}=\left[2\frac{\partial^2\varphi}{\partial r\,\partial z}+2\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\frac{\partial\psi}{\partial r}-\frac{2\psi}{r^2}+k_b{}^2\psi\right]_{z=0}=0 \tag{3}$$

$$u_z\,|_{z=0}=\left[\frac{\partial\varphi}{\partial z}+\frac{\partial\psi}{\partial r}+\frac{\psi}{r}\right]_{z=0}=\begin{cases} v/i\omega & (r<c)\\ 0 & (r>c)\end{cases} \tag{4}$$

Here

$$k_a=\frac{\omega}{a}=\omega\sqrt{\frac{\rho}{\lambda+2\mu}}, \qquad k_b=\frac{\omega}{b}=\omega\sqrt{\frac{\rho}{\mu}}$$

are the wave numbers corresponding to the speeds of propagation of longitudinal and transverse waves a and b (ρ is the density of the medium, and λ and μ are the Lame constants).

The solution of equations (1) and (2) by means of separation of variables yields the following general representation of the bounded solutions:

$$\varphi=\int_0^\infty C_1(\lambda)\,J_0(\lambda r)\exp\left(-z\sqrt{\lambda^2-k_a{}^2}\right)d\lambda \tag{5}$$

$$\psi=\int_0^\infty C_2(\lambda)\,J_1(\lambda r)\exp\left(-z\sqrt{\lambda^2-k_b{}^2}\right)d\lambda \tag{6}$$

where the branches of the roots are chosen to be positive for large values of λ.

The functions $C_1(\lambda)$ and $C_2(\lambda)$ are determined from the boundary conditions. Substitution of (5) and (6) into (3) and (4) yields

$$\int_0^\infty J_1(\lambda r)\left[2\lambda\sqrt{\lambda^2-k_a{}^2}\,C_1(\lambda)+(k_b{}^2-2\lambda^2)\,C_2(\lambda)\right]d\lambda=0 \tag{7}$$

$$\int_0^\infty J_0(\lambda r)\left[-\sqrt{\lambda^2-k_a{}^2}\,C_1(\lambda)+\lambda C_2(\lambda)\right]d\lambda=\begin{cases} v/i\omega & (r<c)\\ 0 & (r>c)\end{cases} \tag{8}$$

Taking into account the discontinuous Weber integral [1]

$$\int_0^\infty J_0(\lambda r)\, J_1(\lambda c)\, d\lambda = \begin{cases} 1/c & (r < c) \\ 0 & (r > c) \end{cases}$$

we see that in order to satisfy equalities (7) and (8) it is sufficient to let

$$2\lambda \sqrt{\lambda^2 + k_a^2} C_1 + (k_b^2 - 2\lambda^2)\, C_2 = 0; \quad -\sqrt{\lambda^2 - k_a^2} C_1 + \lambda C_2 = (vc / i\omega)\, J_1(\lambda c)$$

Then

$$C_2 = \frac{vc}{i\omega} \frac{2\lambda}{k_b^2} J_1(\lambda c), \qquad C_1 = \frac{vc}{i\omega} \frac{2\lambda^2 - k_b^2}{k_b^2 \sqrt{\lambda^2 - k_a^2}} J_1(\lambda c)$$

Thus the solution is given by the following formulas

$$\varphi = \frac{vc}{i\omega} \int_0^\infty \frac{2\lambda^2 - k_b^2}{k_b^2 \sqrt{\lambda^2 - k_a^2}} J_1(\lambda c)\, J_0(\lambda r) \exp\left(-z \sqrt{\lambda^2 - k_a^2}\right) d\lambda \tag{9}$$

$$\psi = \frac{vc}{i\omega} \int_0^\infty \frac{2\lambda}{k_b^2} J_1(\lambda c)\, J_1(\lambda r) \exp\left(-z \sqrt{\lambda^2 - k_b^2}\right) d\lambda \tag{10}$$

In the acoustic case ($k_b \to \infty$) formula (9) gives a new representation of the Rayleigh integral [2] for the displacement potential

$$\varphi = \frac{ivc}{\omega} \int_0^\infty \exp\left(-z \sqrt{\lambda^2 - k^2}\right) J_1(\lambda c)\, J_0(\lambda r) \frac{d\lambda}{\sqrt{\lambda^2 - k^2}} = -\frac{iv}{2\pi\omega} \iint_S \frac{e^{-ivR}}{R}\, dS \tag{11}$$

where R is the distance from a point in the field (r, θ, z) to an element of the piston with coordinates (ρ, ϕ)

$$R = \sqrt{z^2 + r^2 + \rho^2 - 2r\rho \cos(\varphi - \theta)}$$

and the integration is performed over the surface of the piston S.

On the other hand, because of formula (11), one can express the solution (9), (10) in terms of the integral

$$I(k) = \frac{1}{2\pi} \iint_S \frac{e^{-kR}}{R}\, dS$$

as follows:

$$\varphi = \frac{iv}{\omega} \frac{2k_a^2 - k_b^2}{k_b^2} I(k_a) + \frac{iv}{\omega} \frac{2}{k_b^2} \frac{\partial^2}{\partial z^2} I(k_a) \tag{12}$$

$$\psi = \frac{iv}{\omega} \frac{2}{k_b^2} \frac{\partial^2}{\partial z \partial r} I(k_b) \tag{13}$$

Without dwelling on the study of the wave propagation field, whose potentials are given by the formulas (9), (10) or (12), (13), we shall proceed to the more complicated problem of determining the active and reactive resistance of the connection of the piston with the field of propagation. The total mechanical impedance of the piston Z relates the reaction force F of the elastic medium due to the piston to its velocity v

$$F \exp(i\omega t) = Zv \exp(i\omega t)$$

where the force acting on the piston is equal to the integral over the surface of the piston S of the normal stress σ_{zz} in the medium of the surface taken with a reversed sign:

$$F = -\iint\limits_S \sigma_{zz}|_{z\to 0}\, dS \tag{14}$$

Since

$$\sigma_{zz} = \mu\left[2\frac{\partial^2\psi}{\partial r\,\partial z} - \frac{2}{r}\frac{\partial\psi}{\partial z} - 2\frac{\partial^2\varphi}{\partial r^2} - \frac{2}{r}\frac{\partial\varphi}{\partial r} - k_b^2\varphi\right] \qquad \left(\mu = \frac{\omega^2\rho}{k_b^2}\right)$$

and when the values of the potentials (9), (10) and $z = 0$ are substituted here we obtain

$$\sigma_{zz}|_{z\to 0} = -i\omega\rho v c\int_0^\infty J_1(\lambda c)\, J_0(\lambda r)\, \Phi(\lambda)\, d\lambda \tag{15}$$

where

$$\Phi(\lambda) = \frac{(2\lambda^2 - k_b^2)^2}{k_b^4\sqrt{\lambda^2 - k_a^2}} - \frac{4\lambda^2}{k_b^4}\sqrt{\lambda^2 - k_b^2} =$$
$$= \frac{1}{\sqrt{\lambda^2 - k_a^2}} + \frac{4\lambda^2}{k_b^2}\left(\frac{\lambda^2}{k_b^2} - 1\right)\left(\frac{1}{\sqrt{\lambda^2 - k_a^2}} - \frac{1}{\sqrt{\lambda^2 - k_b^2}}\right) \tag{16}$$

From this it can be seen that for $\lambda \to \infty$

$$\Phi(\lambda) \sim 2\frac{k_a^2 - k_b^2}{k_b^4}\lambda$$

which means that the integral (15) is not convergent in the usual sense. The absence of convergence means that the analytical expression for σ_{zz} used is not valid at $z = 0$, which is quite natural with the discontinuous boundary conditions (4) for u_z.

We have to determine the limiting value of σ_{zz} as $z \to 0$. Consequently, equation (15) makes no sense if it is to be understood in terms of convergent integrals. Yet, expression (15) will acquire a completely determined sense if one agrees to study the integration symbol as a limiting value of the convergent integral

$$\int_0^\infty J_1(\lambda c)\, J_0(\lambda r)\, \Phi(\lambda)\, d\lambda = \lim_{\delta\to 0}\int_0^\infty e^{-\delta\lambda} J_1(\lambda c)\, J_0(\lambda r)\, \Phi(\lambda)\, d\lambda,$$

of course, if such a limiting value exists. Such a way of using divergent integrals in computations, originating with Euler and Poisson, is known by the name of the Abel method of summation of divergent integrals [3]. Thus, when using the symbol of the divergent integral we will have in mind operations with a convergent integral up to the limiting operation.

When (15) is substituted into (14) we obtain

$$Z = i\omega\rho c \iint\limits_{S} \int\limits_{0}^{\infty} J_1(\lambda c) J_0(\lambda r) \Phi(\lambda)\, d\lambda\, dS = 2\pi i\omega\rho c \int\limits_{0}^{c} \int\limits_{0}^{\infty} J_1(\lambda c) J_0(\lambda r) \Phi(\lambda)\, d\lambda\, r dr$$

Interchanging the order of integration we will integrate with respect to r:

$$\int\limits_{0}^{c} J_0(\lambda r)\, r dr = \frac{c}{\lambda} J_1(\lambda c)$$

When

$$Z = 2\pi i\omega\rho c^2 \int\limits_{0}^{\infty} J_1^2(\lambda c) \Phi(\lambda) \frac{d\lambda}{\lambda}$$

the function (16) is represented in the form

$$\Phi(\lambda) = \frac{1}{k_b^4} [(2k_a^2 - k_b^2)^2 (\lambda^2 - k_a^2)^{-1/2} + 4(2k_a^2 - k_b^2)(\lambda^2 - k_a^2)^{1/2} - \\ - 4_b^2 (\lambda^2 - k_b^2)^{1/2} + 4(\lambda^2 - k_a^2)^{3/2} - 4(\lambda^2 - k_b^2)^{3/2}]$$

and the notation

$$A_n(k) = \int\limits_{0}^{\infty} J_1^2(\lambda c)(\lambda^2 - k^2)^{n-1/2} \frac{d\lambda}{\lambda} \tag{17}$$

is introduced, where the integral sign is to be understood in the sense of the Abel summation, then we shall obtain

$$Z = 2\pi i\omega\rho c^2 \left[\frac{(2k_a^2 - k_b^2)^2}{k_b^4} A_0(k_a) + \frac{4(2k_a^2 - k_b^2)}{k_b^4} A_1(k_a) - \right. \\ \left. - \frac{4}{k_b^2} A_1(k_b) + \frac{4}{k_b^4} A_2(k_a) - \frac{4}{k_b^4} A_2(k_b) \right] \tag{18}$$

Now we shall evaluate the integrals (17). First of all let us perform a change of variable

$$\lambda c = u \qquad (kc = v) \tag{19}$$

then

$$A_n(k) = \frac{1}{c^{2n-1}} \int\limits_{0}^{\infty} J_1^2(u)(u^2 - v^2)^{n-1/2} \frac{du}{u}$$

Furthermore, using the Neumann integral,

$$J_1^2(u) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} J_2(2u\sin\theta)\,d\theta$$

one can write

$$A_n(k) = \frac{2}{\pi c^{2n-1}}\int_0^{\infty}\int_0^{\frac{1}{2}\pi} J_2(2u\sin\theta)(u^2-v^2)^{n-\frac{1}{2}}\frac{d\theta\,du}{u}$$

With the aid of the recurrence formula

$$J_2(2u\sin\theta) = \frac{u\sin\theta}{2}[J_1(2u\sin\theta)+J_3(2u\sin\theta)]$$

written in terms of Bessel integrals

$$J_2(2u\sin\theta) = \frac{u\sin\theta}{\pi}\int_0^{\frac{1}{2}\pi}(\sin\varphi+\sin 3\varphi)\sin(2u\sin\theta\sin\varphi)\,d\varphi$$

we obtain

$$A_n(k) = \frac{8}{\pi^2 c^{2n-1}}\int_0^{\infty}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin\varphi\cos^2\varphi\sin\theta\sin(2u\sin\varphi\sin\theta)(u^2-v^2)^{n-\frac{1}{2}}\,d\varphi\,d\theta\,du \tag{20}$$

Now we shall interchange the order of integration and the internal integral with respect to the variable and sum according to Abel. Then the integral becomes

$$B_n = \int_0^{\infty}\sin(2u\sin\varphi\sin\theta)(u^2-v^2)^{n-\frac{1}{2}}du = \qquad (x = 2v\sin\varphi\sin\theta)$$

$$= -i(-1)^n v^{2n}\int_0^1\sin(xt)(1-t^2)^{n-\frac{1}{2}}dt + v^{2n}\int_1^{\infty}\sin(xt)(t^2-1)^{n-\frac{1}{2}}dt \tag{21}$$

The integral with finite limits is an ordinary integral. It determines the Struve function with the index n

$$\int_0^1\sin(xt)(1-t^2)^{n-\frac{1}{2}}dt = \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\left(\frac{1}{2}x\right)^n}\mathbf{H}_n(x) \tag{22}$$

The second integral is computed using the Schlafli representation

$$\int_0^{\infty} e^{-zt}(t^2-1)^{n-\frac{1}{2}}dt = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\left(\frac{z}{2}\right)^n}K_n(z) \qquad \left(z = \delta - ix, \delta > 0, n > -\frac{1}{2}\right)$$

When the limit

$$\lim_{\delta\to 0}\int_1^{\infty} e^{-zt}(t^2-1)^{n-\frac{1}{2}}dt = \lim_{z\to -ix}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\left(\frac{z}{2}\right)^n}K_n(z) =$$

$$= \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\left(-\frac{ix}{2}\right)^n} K_n(-ix) = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)^n} \frac{(-1)^n}{i^n} \frac{\pi i}{2} \exp\frac{n\pi i}{2} [J_n(x) + iY_n(x)]$$

exists then we obtain

$$\int_1^\infty \sin(xt)(t^2-1)^{n-1/2}\,dt = \lim_{\delta\to 0}\int_1^\infty e^{-\delta t}\sin(xt)(x^2-1)^{n-1/2}\,dt =$$

$$= \lim_{\delta\to 0} Im\int_1^\infty e^{-zt}(t^2-1)^{n-1/2}\,dt = \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\left(\frac{x}{2}\right)^n}(-1)^n J_n(x) \qquad (23)$$

From this it follows that the generalized Mehler-Sonine formula [1]

$$J_\nu(x) = \frac{2}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)^\nu}\int_1^\infty \frac{\sin(xt)\,dt}{(t^2-1)^{\nu+1/2}}$$

is correct in the sense of the Abel summation not only for the values $-1/2 < \nu < 1/2$ but also for all negative values of ν.

When formulas (22) and (23) are combined we obtain the following expression for the integral (21):

$$B_n = \frac{(-1)^n\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)v^{2n}}{2\left(\frac{\alpha\sin\theta}{2}\right)^n}[J_n(\alpha\sin\theta) - i\mathbf{H}_n(\alpha\sin\theta)] \qquad (\alpha = 2v\sin\varphi)$$

The remaining integrals in expression (20) have the usual meaning. When (19) is taken into account this expression becomes:

$$A_n(k) = \frac{(-1)^n 4}{\pi^2}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)k^{2n}c\times$$

$$\times\int_0^{1/2\pi}\sin\varphi\cos^2\varphi\int_0^{1/2\pi}[J_n(\alpha\sin\theta) - i\mathbf{H}_n(\alpha\sin\theta)]\left(\frac{\alpha\sin\theta}{2}\right)^{-n}\sin\theta\,d\theta\,d\varphi \qquad (24)$$

The integration with respect to θ will be accomplished by expanding the Bessel and Struve functions into power series:

$$C_n(\varphi) = \int_0^{1/2\pi}[J_n(\alpha\sin\theta) - i\mathbf{H}_n(\alpha\sin\theta)]\left(\frac{\alpha\sin\theta}{2}\right)^{-n}\sin\theta\,d\theta =$$

$$= \int_0^{1/2\pi}\left(\frac{\alpha\sin\theta}{2}\right)^{-n}\sin\theta\sum_{m=0}^\infty\frac{\exp-\frac{m\pi i}{2}}{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{m}{2}+n+1\right)}\left(\frac{\alpha\sin\theta}{2}\right)^{n+m}d\theta =$$

$$= \sum_{m=0}^\infty\frac{\exp-\frac{m\pi i}{2}}{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{m}{2}+n+1\right)}\left(\frac{\alpha}{2}\right)^m\int_0^{1/2\pi}\sin^{m+1}\theta\,d\theta =$$

$$= \sum_{m=0}^{\infty} \frac{\exp - \frac{m\pi i}{2}}{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{m}{2}+n+1\right)} \left(\frac{\alpha}{2}\right)^m \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m}{2}+1\right)}{2\Gamma\left(\frac{m+1}{2}+1\right)}$$

Then

$$C_n(\varphi) = \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}\right)}{2} \frac{\exp - \frac{m\pi i}{2}}{\Gamma\left(\frac{m+1}{2}+1\right)\Gamma\left(\frac{m}{2}+n+1\right)} \left(\frac{\alpha}{2}\right)^m$$

Now one can compute the last integral with respect of ϕ:

$$D_n(k) = \int_0^{\frac{1}{2}\pi} \sin\varphi \cos^2\varphi C_n(\varphi)\, d\varphi =$$

$$= \int_0^{\frac{1}{2}\pi} \sin\varphi \cos^2\varphi \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}\right)}{2} \frac{\exp - \frac{m\pi i}{2}}{\Gamma\left(\frac{m+1}{2}+1\right)\Gamma\left(\frac{m}{2}+n+1\right)} \left(\frac{\alpha}{2}\right)^m d\varphi =$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}\right)}{2} \frac{\exp\left(-\frac{m\pi i}{2}\right) v^m}{\Gamma\left(\frac{m+1}{2}+1\right)\Gamma\left(\frac{m}{2}+n+1\right)} \int_0^{\frac{1}{2}\pi} \sin^{m+1}\varphi \cos^2\varphi\, d\varphi =$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}\right)}{2} \frac{\exp\left(-\frac{m\pi i}{2}\right) v^m}{\Gamma\left(\frac{m+1}{2}+1\right)\Gamma\left(\frac{m}{2}+n+1\right)} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{m}{2}+1\right)}{2\Gamma\left(\frac{m+1}{2}+2\right)}$$

which can be written in the following way:

$$D_n(k) = \frac{\pi}{8} \sum_{m=0}^{\infty} \frac{\exp\left(-\frac{m\pi i}{2}\right) v^m 2^n}{\Pi(l)\,\Gamma\left(\frac{m+1}{2}+1\right)\Gamma\left(\frac{m+1}{2}+2\right)} \qquad \left(\Pi(l) = \prod_{l=1}^{n} (m+2l)\right) \tag{25}$$

It can easily be seen that

$$\frac{d}{dv}\left(v^{2n} D_n\right) = \frac{\pi}{8} \sum_{m=0}^{\infty} \frac{\exp\left(-\frac{m\pi i}{2}\right) v^{m+2n-1} 2^n (m+2n)}{\Pi(l)\,\Gamma\left(\frac{m+1}{2}+1\right)\Gamma\left(\frac{m+1}{2}+2\right)} = 2v^{2n-1} D_{n-1}$$

From this, assuming that D_n is a function which is complete and bounded at zero, one obtains a recurrence formula

$$D_n = \frac{2}{v^{2n}} \int_0^v v^{2n-1} D_{n-1}\, dv \tag{26}$$

For $n = 0$ the series (25) becomes

$$D_0(k) = \frac{\pi}{8} \sum_{m=0}^{\infty} \frac{\exp\left(-\frac{m\pi i}{2}\right) v^m}{\Gamma\left(\frac{m+1}{2}+1\right)\Gamma\left(\frac{m+1}{2}+2\right)} =$$

$$= \frac{\pi}{8v^2} \sum_{p=1}^{\infty} \frac{\exp\frac{i\pi}{2} \exp\left(-\frac{p\pi i}{2}\right) v^{p+1}}{\Gamma\left(\frac{p}{2}+1\right)\Gamma\left(\frac{p}{2}+1+1\right)} = -\frac{\pi i}{8v}\left[1 - \frac{J_1(2v)}{v} + i\,\frac{\mathbf{H}_1(2v)}{v}\right]$$

The recurrence formula yields

$$D_1(k) = -\frac{\pi i}{4v}\left[1 - \frac{1}{v}\int_0^{2v} \frac{J_1(x)}{x}\,dx + \frac{i}{v}\int_0^{2v} \frac{\mathbf{H}_1(x)}{x}\,dx\right] = -\frac{\pi i}{4v}\left[1 - \frac{\bar{J}(2v) - J_1(2v)}{v} + i\,\frac{\bar{\mathbf{H}}(2v) - \mathbf{H}_1(2v)}{v}\right] \qquad (28)$$

where

$$\bar{J}(x) = \int_0^x J_0(\xi)\,d\xi, \qquad \bar{\mathbf{H}}(x) = \int_0^x \mathbf{H}_0(\xi)\,d\xi$$

are tabulated functions. Similarly

$$D_2(k) = -\frac{\pi i}{6v}\left\{1 - \frac{3}{v^3}\int_0^v [\bar{J}(2v) - J_1(2v)]\,v\,dv + \frac{3i}{v^3}\int_0^v [\bar{\mathbf{H}}(2v) - \mathbf{H}_1(2v)]\,v\,dv\right\} =$$

$$= -\frac{\pi i}{6v}\left\{1 - 3\left[\frac{\bar{J}(2v) - J_1(2v)}{2v} + \frac{J_0(2v)}{4v^2} - \frac{\bar{J}(2v)}{8v^3}\right] + 3i\left[\frac{\bar{\mathbf{H}}(2v) - \mathbf{H}_1(2v) - 1/\pi}{2v} + \frac{\mathbf{H}_0(2v)}{4v^2} - \frac{\bar{\mathbf{H}}(2v)}{8v^3}\right]\right\}$$

Now expression (24) becomes:

$$A_0(k) = \frac{4c}{\pi} D_0 = -\frac{i}{2k}\left[1 - \frac{J_1(2kc)}{kc} + i\,\frac{\mathbf{H}_1(2kc)}{kc}\right]$$

$$A_1(k) = -\frac{2k^2c}{\pi} D_1 = \frac{ik}{2}\left[1 - \frac{\bar{J}(2kc) - J_1(2kc)}{kc} + i\,\frac{\bar{\mathbf{H}}(2kc) - \mathbf{H}_1(2kc)}{kc}\right]$$

$$A_2(k) = \frac{3k^4c}{\pi} D_2 = -\frac{ik^3}{2}\left\{1 - 3\left[\frac{\bar{J}(2kc) - J_1(2kc)}{2kc} + \frac{J_0(2kc)}{(2kc)^2} - \frac{\bar{J}(2kc)}{(2kc)^3}\right] + 3i\left[\frac{\bar{\mathbf{H}}(2kc) - \mathbf{H}_1(2kc) - 1/\pi}{2kc} + \frac{\mathbf{H}_0(2kc)}{(2kc)^2} - \frac{\bar{\mathbf{H}}(2kc)}{(2kc)^3}\right]\right\}$$

Substitution of these functions into formula (18) yields the following final formulas for the resistance of the driving of the piston

$$Z = S\rho a\,[R(k_a c;\, k_b c) + iX(k_a c,\, k_b c)]$$

where $S = \pi c^2$ is the area of the piston, ρ is the density of the medium, a is the speed of propagation of the lingitudinal waves, c is the radius of the piston, and k_a and k_b are the wave numbers corresponding to speeds of propagation of longitudinal and transverse waves. The dimensionless active resistance R and the reactive resistance X of the connection of the piston with the wave propagation field is expressed in terms of tabulated functions in the following fashion:

$$R = 1 - \left[1 - 8\left(\frac{b}{a}\right)^2 + 6\left(\frac{b}{a}\right)^4\right]\frac{J_1(2k_a c)}{k_a c} - 2\left(\frac{b}{a}\right)\frac{J_1(2k_b c)}{k_b c} - \tag{29}$$
$$-\left[4\left(\frac{b}{a}\right)^2 - 2\left(\frac{b}{a}\right)^4\right]\frac{\bar{J}(2k_a c)}{k_a c} + 2\left(\frac{b}{a}\right)\frac{\bar{J}(2k_b c)}{k_b c} -$$
$$-3\left(\frac{b}{a}\right)^4\frac{J_0(2k_a c)}{(k_a c)^2} + \frac{3}{2}\left(\frac{b}{a}\right)^4\frac{\bar{J}(2k_a c)}{(k_a c)^3} + 3\left(\frac{b}{a}\right)\frac{J_0(2k_b c)}{(k_b c)^2} - \frac{3}{2}\left(\frac{b}{a}\right)\frac{\bar{J}(2k_b c)}{(k_b c)^3}$$

$$X = \left[1 - 8\left(\frac{b}{a}\right)^2 + 6\left(\frac{b}{a}\right)^4\right]\frac{\mathbf{H}_1(2k_a c)}{k_a c} + 2\left(\frac{b}{a}\right)\frac{\mathbf{H}_1(2k_b c)}{k_b c} + \tag{30}$$
$$+\left[4\left(\frac{b}{a}\right)^2 - 2\left(\frac{b}{a}\right)^4\right]\frac{\bar{\mathbf{H}}(2k_a c)}{k_a c} - 2\left(\frac{b}{a}\right)\frac{\bar{\mathbf{H}}(2k_b c)}{k_b c} +$$
$$+3\left(\frac{b}{a}\right)^4\frac{\mathbf{H}_0(2k_a c)}{(k_a c)^2} - \frac{3}{2}\left(\frac{b}{a}\right)^4\frac{\bar{\mathbf{H}}(2k_a c)}{(k_a c)^3} - \frac{6}{\pi}\left(\frac{b}{a}\right)^4\frac{1}{k_a c} -$$
$$-3\left(\frac{b}{a}\right)\frac{\mathbf{H}_0(2k_b c)}{(k_b c)^2} + \frac{3}{2}\left(\frac{b}{a}\right)\frac{\bar{\mathbf{H}}(2k_b c)}{(k_b c)^3} + \frac{6}{\pi}\left(\frac{b}{a}\right)\frac{1}{k_b c}$$

In the case of a fluid ($b = 0$), formulas (29) and (30) become the well-known Rayleigh formulas.

The above dependence of the mechanical impedance of a piston with a rigid collar upon the parameters ρ, λ and μ can be utilized, for instance, for the determination of the physical properties of elastic bodies (in particular rock strata), just as the analysis of the impedance of a piston acting upon a fluid flow [4,5] can be used for the measurement of the flow velocity.

Formulas (29) and (30) are easily studied in the limiting cases. For wavelengths that are small compared to the dimensions of the driver one has $k_a c \gg 1$ and $k_b c \gg 1$. In this case the reactive resistance is small and the dimensionless active resistance approaches unity, so that

$$Z \approx S\rho a$$

which means that the character of the driving is the same as in the case of a high-frequency driving of a fluid of uniform density moving at the speed of sound, equal to the speed of propagation of longitudinal waves.

At low frequencies, when $k_a c \ll 1$ and $k_b c \ll 1$ we obtain

$$X \approx k_a c\left[\frac{4}{\pi} - \frac{16}{3\pi}\left(\frac{b}{a}\right)^2 + \frac{4}{\pi}\left(\frac{b}{a}\right)^4\right]$$
$$R = (k_a c)^2\left[\frac{1}{2} + \frac{4}{15}\left(\frac{a}{b}\right) - \frac{4}{3}\left(\frac{b}{a}\right)^2 + \frac{16}{15}\left(\frac{b}{a}\right)^4\right]$$

To avoid misunderstanding let us note that in the last formulas the transition to the case of a fluid ($k_b \to \infty$) is impossible since they were obtained under the condition that $k_b c \ll 1$, i.e. for media in which the speed of propagation of transverse waves was comparable to the speed of longitudinal waves.

TABLE 1.

$k_a c$	$R(kc)$		$X(kc)$	
	Elastic Poisson medium	Ideal fluid	Elastic Poisson medium	Ideal fluid
0.25	0.0376	0.0309	0.2063	0.2087
0.50	0.1461	0.1199	0.3772	0.3969
0.75	0.2960	0.2561	0.4891	0.5471
1.00	0.4560	0.4233	0.5345	0.6468
1.25	0.5967	0.6023	0.5232	0.6905
1.50	0.7005	0.7740	0.4761	0.6801
1.75	0.7639	0.9215	0.4184	0.6238
2.00	0.7956	1.0330	0.3675	0.5349
2.25	0.8101	1.1027	0.3323	0.4293
2.50	0.8204	1.1310	0.3188	0.3231
2.75	0.8337	1.1242	0.2993	0.2300
3.00	0.8508	1.0922	0.2875	0.1594
3.25	0.8681	1.0473	0.2724	0.1159
3.50	0.8822	1.0013	0.2543	0.0989
3.75	0.8882	0.9639	0.2365	0.1036
4.00	0.8902	0.9413	0.2226	0.1220

The analysis using formulas (29) and (30) is also sufficiently simple even in those cases when it becomes necessary to enlarge the extent or the accuracy of the existing tables of integrals

$$\bar{J}(x) = \int_0^x J_0(\xi)\, d\xi, \quad \bar{\mathrm{H}}(x) = \int_0^x \mathrm{H}_0(\xi)\, d\xi$$

As an illustration we present in Table 1 a computation of the values of the dimensionless active and reactive resistance using the formulas (29) and (30) for the case of driving of a Poisson medium ($\lambda = \mu$). For comparison also the corresponding values for the case of the fluid [6] are presented. The argument is kc, wherein the wave number corresponds to the value of the speed of longitudinal waves.

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